TESTING… TESTING…

TESTING FOR CONVERGENCE INCLUDING AT ENDPOINTS

What Could Happen?

1)The series could diverge

2)The series could converge absolutely

3)The series could converge conditionally

The n-th Term Test

PROS – Quick test, should be your first try
CONS – Doesn't always work

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ **fails to exist** or is **different from zero.**

If $\lim_{n\to\infty} a_n = 0$, then the series MAY converge, or it MAY diverge.

Determine the convergence or divergence of the series shown.

Ex 1: $\sum_{n=1}^{\infty} \frac{n^2+2}{n^2+1}$

Ex 2: $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{2n^2+5}$

Ex 3: $\sum_{n=1}^{\infty} \ln(\cos(n))$

THE INTEGRAL TEST

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N is a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ either both converge or both diverge.

In other words, if the integral has a value, then it must converge, and likewise, the series must then converge!

THE INTEGRAL TEST

Some Examples

Do the series below converge?

Ex 1:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Ex 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

<u>Does the I.T. apply?</u>

Is f continuous?

Is f positive?

Is f decreasing?

CAUTION!!!

The series and the integral in the Integral Test need not have the same value in the convergent case. Although the integral converges to $\dfrac{\pi}{4}$ in Example 2, the series might have a quite different sum!

Also, the unfortunate part of all these tests is that the tests only help determine if a series converges – not what it sums to!

THE P-SERIES TEST

Occasionally, we have come across series such as these: $\sum_{1}^{\infty} \frac{1}{n^p}$. Such a series is called a "p-series".

(1) Use the Integral Test to prove that $\sum_{1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

(2) Use the Integral Test to prove that $\sum_{1}^{\infty} \frac{1}{n^p}$ diverges if $p < 1$.

(3) Use the Integral Test to prove that $\sum_{1}^{\infty} \frac{1}{n^p}$ diverges if $p = 1$.

Exploring Endpoint Convergence

For what values of x does the series below converge?

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

(1) Apply the Ratio Test to determine the radius of convergence.

(2) Substitute the right-hand endpoint of the interval into the power series. You should get:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n-1}}{n} + \ldots$$

(3) Chart the progress of the partial sums of this series geometrically on a number line as follows: Start at 0, go forward 1. Go back ½ . Go forward 1/3. Go back ¼ . Go forward 1/5, and so on.

(4) Does the series converge at the right-hand endpoint? Give a convincing argument based on your geometric journey in part 3?

(5) Does the series converge absolutely at the right-hand endpoint?

The Direct Comparison Test
(Otherwise called the Limit Comparison Test)

Suppose that $a_n > 0$ *and* $b_n > 0$ for all $n \geq N$ (*N* is a positive integer)

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c,\ 0 < c < \infty,$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

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Examples

Determine the convergence or divergence of the series shown.

Ex 1: $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$ Case I

Ex 2: $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ Case II

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A Hard Example

Determine the convergence or divergence of the series shown.

Case II…again Ex 3: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Alternating Series

Here's what an alternating series looks like…

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{(-1)^{n+1}}{n}+\ldots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n}4}{2^{n}}+\ldots$$

$$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$$

So how do we prove the convergence of such series?

THE ALTERNATING SERIES TEST

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$

converges if BOTH of the following conditions are satisfied:

1. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N,
2. $\lim_{x\to\infty} u_n = 0$

ALTERNATING SERIES

Examples

Determine the convergence or divergence of the series shown.

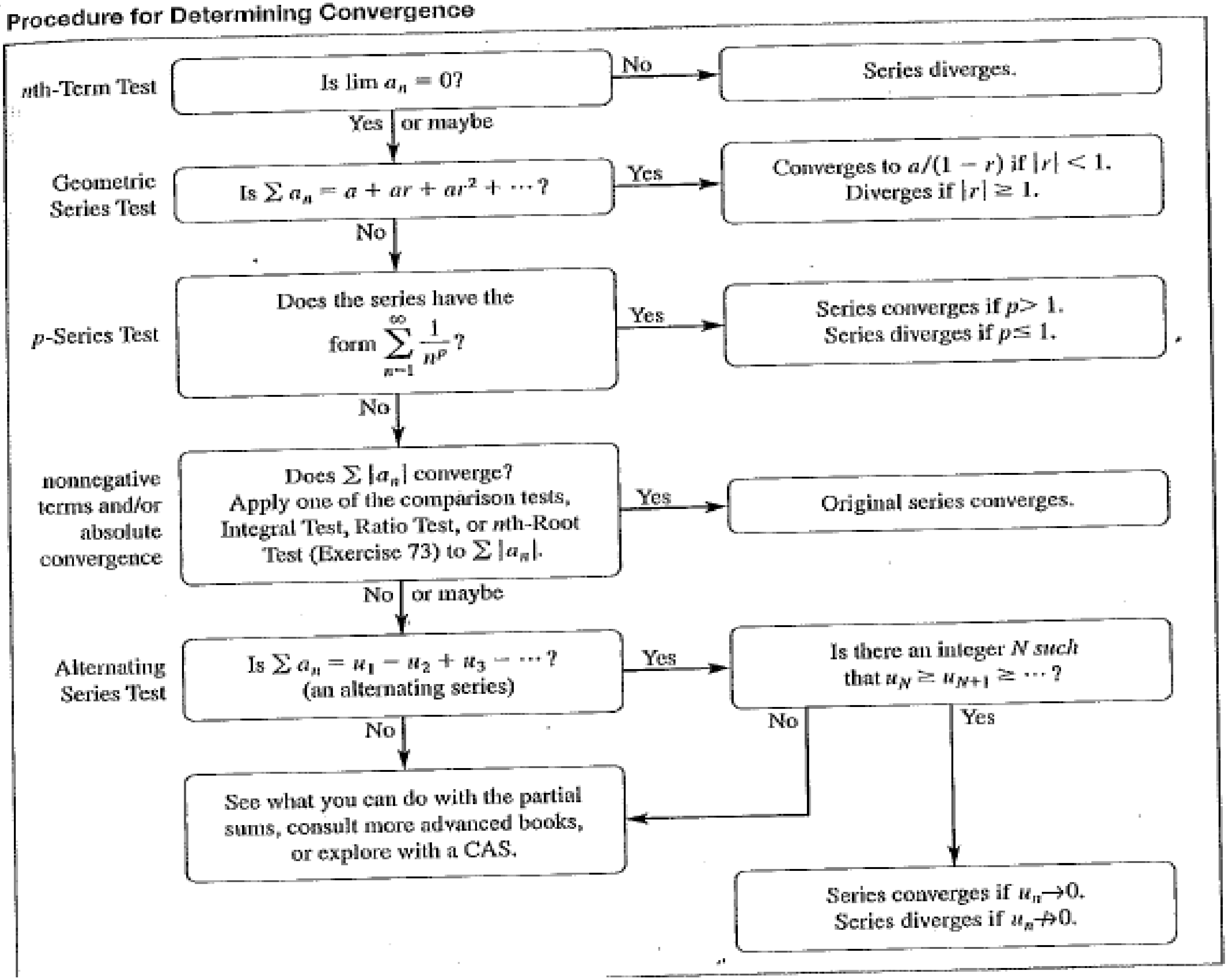
Ex 1:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

Ex 2:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{n^{10}}$$

Procedure for Determining Convergence



AP Exam MC

(No Calculator)

The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

(A) $-4 \le x < -2$
(B) $-1 < x < 1$
(C) $-1 \le x < 1$
(D) $2 < x < 4$
(E) $2 \le x < 4$

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No Calculator

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

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Answers

(a) $\ln\left(\dfrac{1}{1+3x}\right) = \ln\left(\dfrac{1}{1-(-3x)}\right)$

$$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n} \text{ or } \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\dfrac{1}{3} < x \le \dfrac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

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Answers

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.	$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$
(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.	$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$