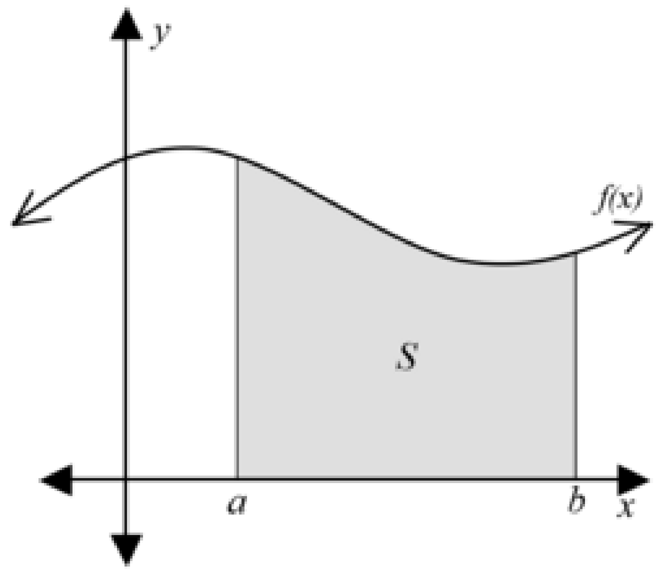
Area Under the Curve



Riemann Sums

And the Trapezoidal Rule

Who knew that $D=R \times T$ would connect to velocity, and now integration, and the area under a curve?

Take a look at the attached applications. Let's see how this simple concept can take on such fascinating aspects of calculus!!!

Graph Word Problems

A person is on the highway. The car they are driving is moving at the rate of 65 mph. At 1:00, the driver presses the cruise control on the car and then lets the car move down the highway at this steady rate for 4 hours.

a) **Draw the graph of this problem for the 4 hours.**

b) **Find the area of the region under this curve.**

c) **Find the slope of the curve.**

d) **What do each of these numbers mean?**

e) **How far did they travel in the 4 hours?**

Graph Word Problems

A person is on a highway at 65 mph for 2 hours. Suddenly, there is a rain storm and the driver immediately hits his brakes and the car is now moving at 45 mph for the next 4 hours.

a) Draw the graph of this problem .

b) Find the area under the curve for the first 2 hours.
c) Find the area under the curve for the next 4 hours.
d) Find the total distance that the driver covered in the 6 hours spent on the road.

Graph Word Problems

A driver is going from home to work. It is a straight path from his home to work with no stops along the way. It takes him 2 minutes to get to 40 mph. He then travels at 40 mph for 20 minutes and then it takes him another 2 minutes to come to a stop.

a) Draw the graph of this problem.

b) Find the area under the curve.

c) Find the distance traveled.

d) Find the slope of the first two minutes and the last two minutes. What do these numbers represent?

Estimating Area

Consider 3 methods:

(A)LRAM - Left-endpoint Rectangular Approximation

(Left Riemann Sum)

(B) MRAM - Midpoint Rectangular Approximation

(Midpoint Riemann Sum)

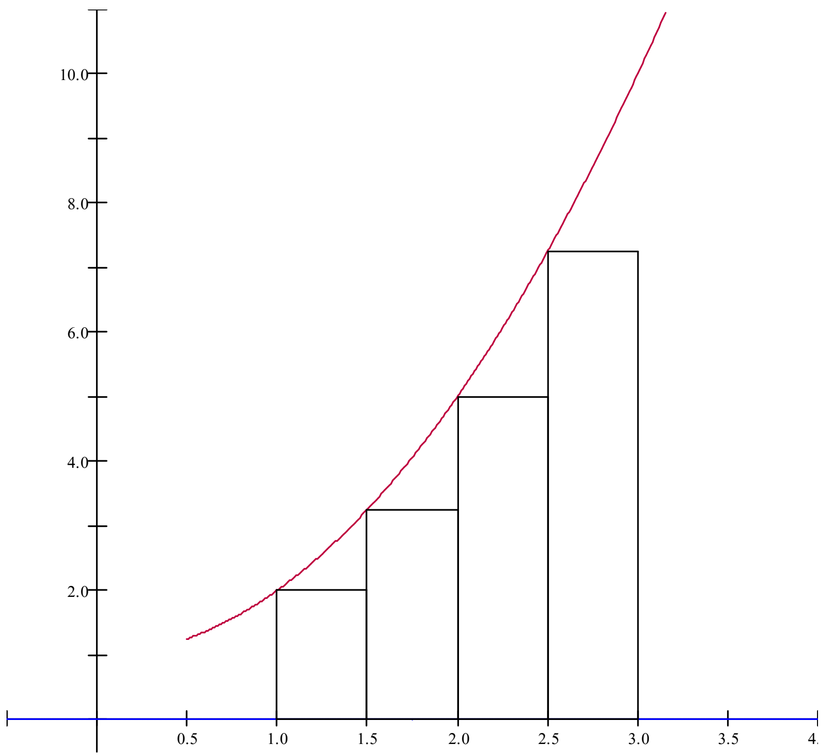
(C) RRAM - Right-endpoint Rectangular Approximation

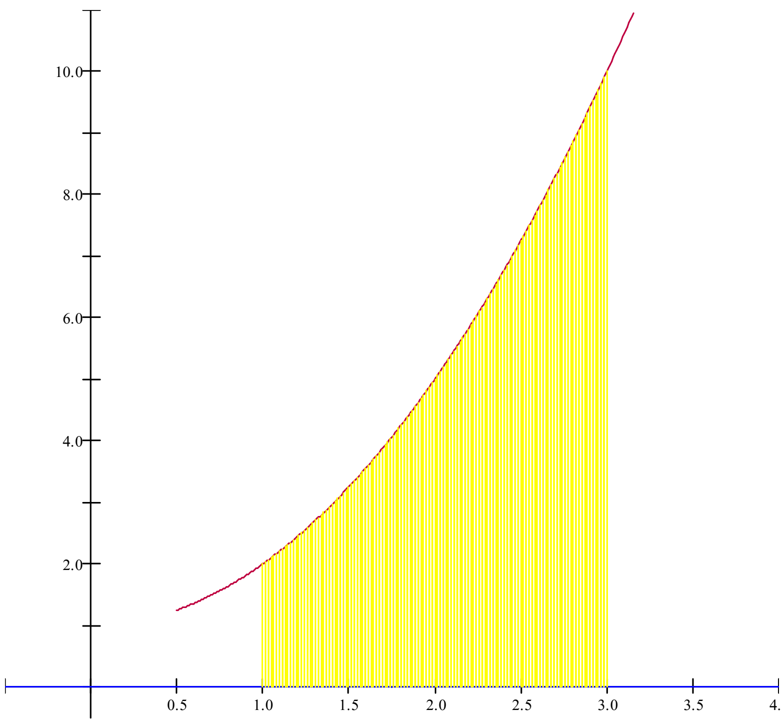
(Right Riemann Sum)

LRAM – LEFT RIEMANN SUM

Example: *Find the area bounded by the curve* $f(x) = x^2 + 1, on\ [1,3]$

using 4 rectangles of equal width.

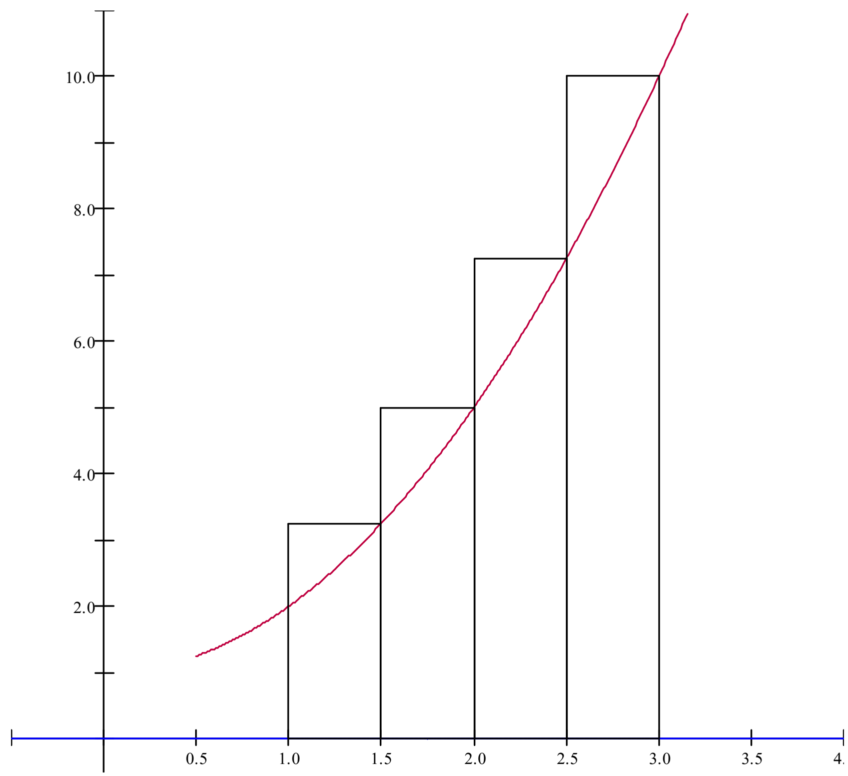




Would this approximation be an overestimate or underestimate?

RRAM – RIGHT RIEMANN SUM

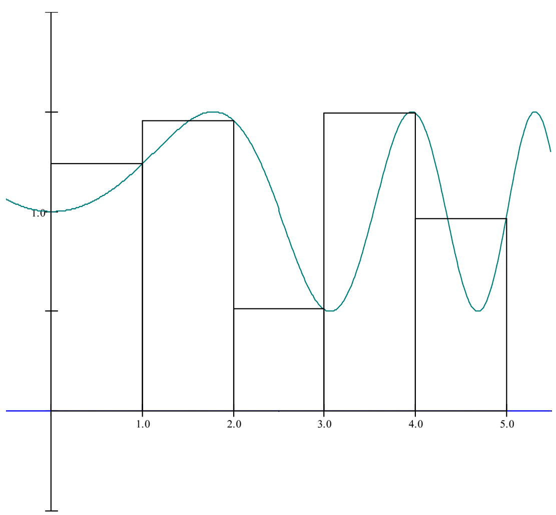
Example: *Find the area bounded by the curve* $f(x) = x^2 + 1, on\ [1,3]$ *using 4 rectangles of equal width.*



Would this approximation be an overestimate or underestimate?

Although we could establish rules to determine whether a left or right Riemann sum for an increasing or decreasing function will overestimate or underestimate the actual area, it is much more instructive to use a sketch of the graph to reach a conclusion. It is also important to note that CONCAVITY DOES NOT AFFECT THIS ISSUE like it did for tangent lines.

Furthermore, there are many cases where it is not clear whether an approximation exceeds or falls short of the actual value. For example, a right Riemann sum applied to the curve shown below leaves us with an unclear picture about whether the sum overestimates or underestimates the actual area.

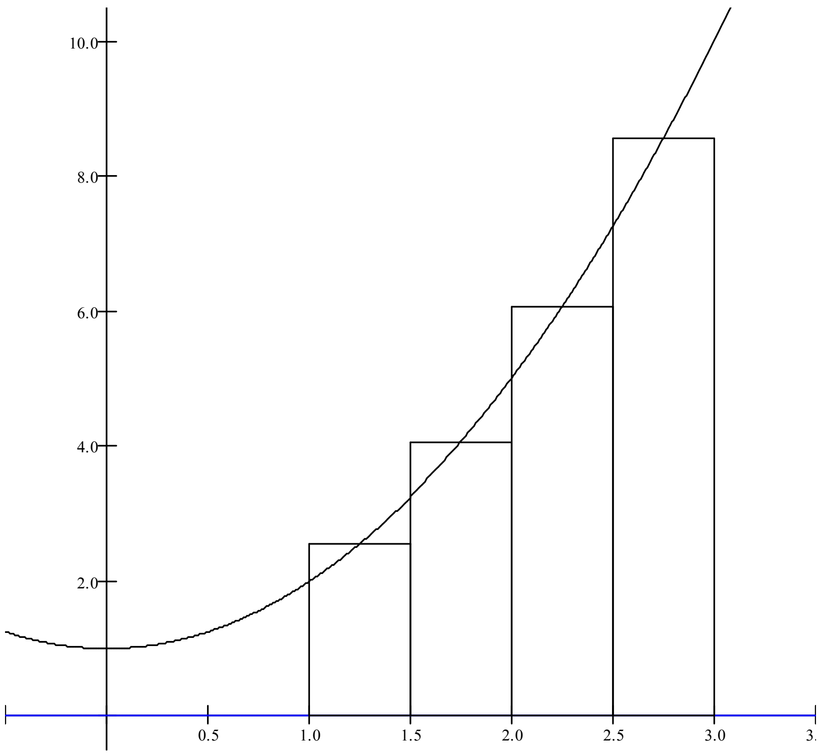


$$y = .5\sin(.5x^2) + 1$$

MRAM – Midpoint Riemann Sum

Example: *Find the area bounded by the curve* $f(x) = x^2 + 1 \text{ on } [1,3]$ *using 4 rectangles of equal width.*

This is often the preferred method of estimating area because it tends to balance overage and underage - look at the space between the rectangles and the curve as well as the amount of rectangle space above the curve and this becomes more evident.



Assumption Alert!

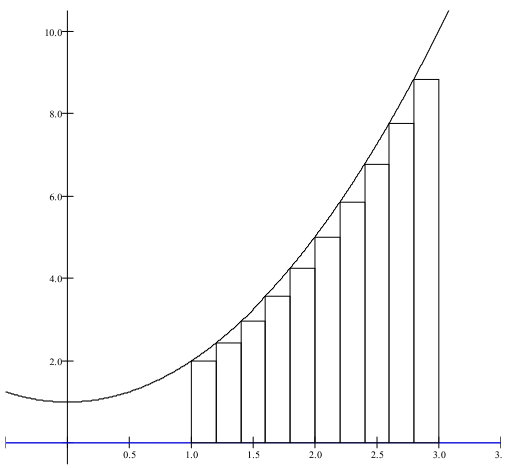
Students often mistakenly believe that this balance is perfect and that the midpoint approximation is exact. In other words, that the MRAM is simply the average of the LRAM and RRAM.

NOT TRUE!!!

Let's verify this…..

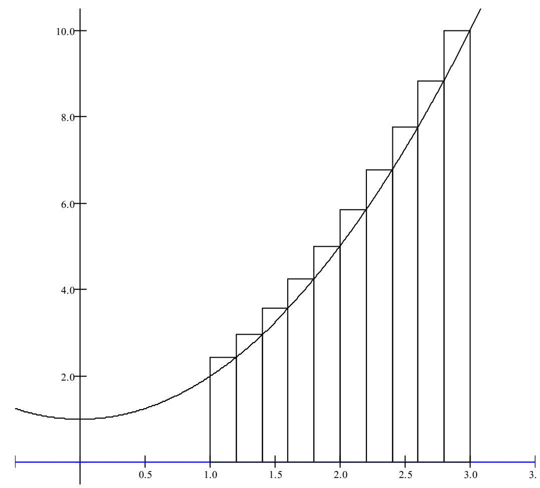
What happens if we add more rectangles?

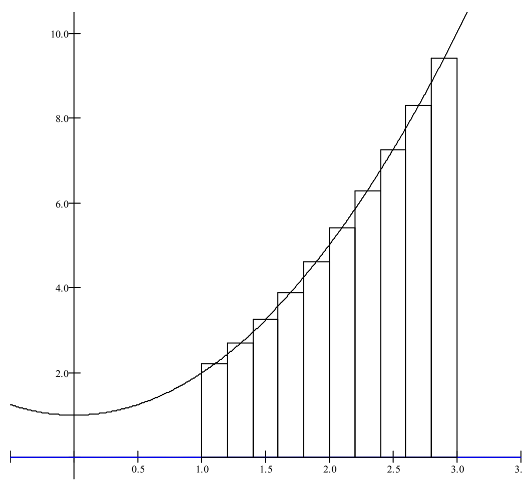
Example 2: Find the area bounded by the curve $f(x) = x^2 + 1$ Using 10 rectangles of equal width.



LRAM = 9.88

RRAM = 11.48

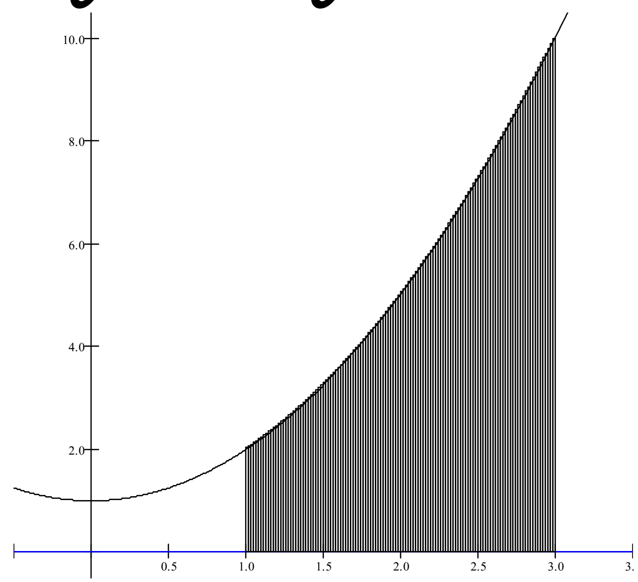




MRAM = 10.66

What would lead us to obtaining the best possible estimate (perhaps exact) area?

Using Infinitely Many Rectangles!!!

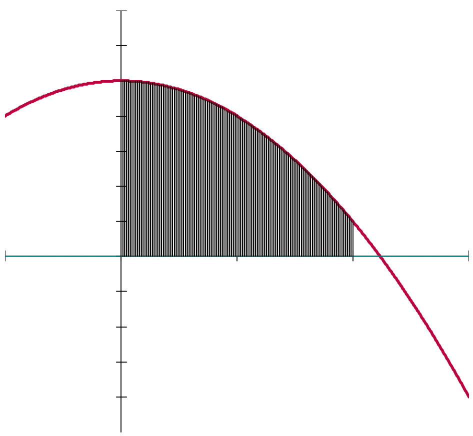


This view shows the area under the curve using 1000 rectangles!! Pretty exact, huh?

You Try!!!

Ex 2: Find the area bounded by the curve $f(x) = -x^2 + 5, on[0,2]$

Use 5 rectangles of equal width, and compute LRAM, MRAM, and RRAM.



Did you notice anything peculiar when you computed your areas, as compared to our previous example???

Let's Explore

Which RAM is the biggest?

See the attached exploration, and let's find out what influences these approximations!?!?!?!

Which Geometric Shape Might Provide a BETTER Approximation?

That's Right! A Trapezoid

whose area is…

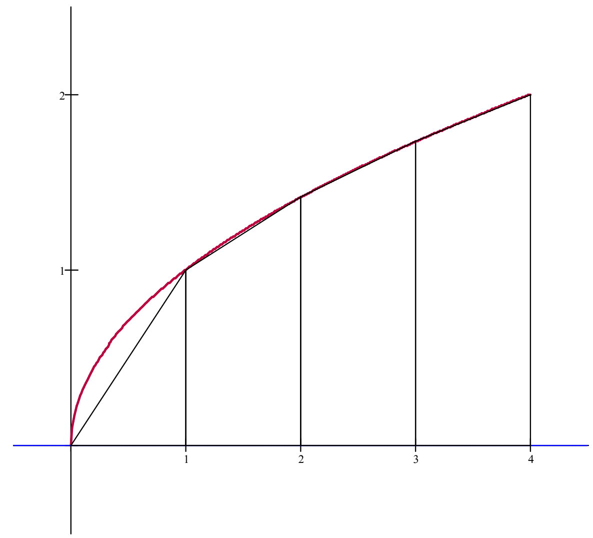
$$A = \frac{1}{2}h(b_1 + b_2)$$

Some Examples

Non-Calculator

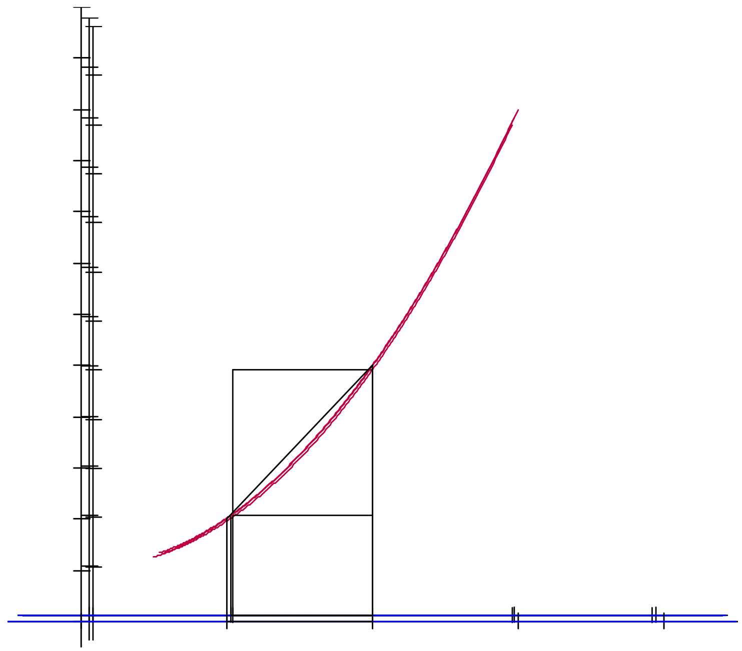
Approximate the area under the given curve on the interval [0, 4] using trapezoids and 4 equal subintervals.

Ex 1: $f(x) = \sqrt{x}$



Just for fun, let's also calculate the LRAM and RRAM!

Is there a *FASTER* way?



Further Examples

with Calculator

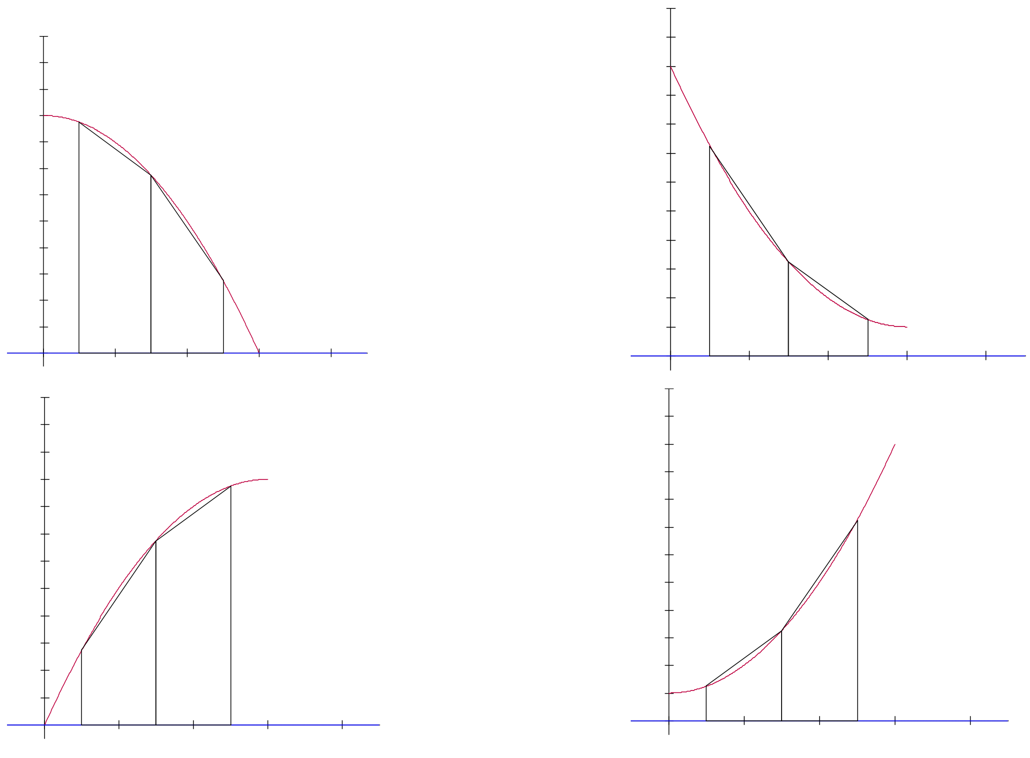
Approximate the area under the given curve on the interval given using trapezoids and 4 equal subintervals.

Ex 1: $f(x) = \cos x$

$$\left[0, \frac{\pi}{3}\right]$$

Overestimate or Underestimate?

Initially, and given our Riemann sum discussion, many people assume that whether the function increases or decreases affects the size of the approximation relative to the actual value…..but it doesn't seem like this is true does it?



What feature affects our approximation?

Equal Subintervals

ACTUARIES
IN
ACTION

Equal Subintervals

CARDIOLOGY

The following table gives the dye concentration for a dye-concentration cardiac-output determination seconds after injection. The amount of dye injected in this patient was 5 mg. Use a left-point Riemann sum to estimate the area under the dye concentration curve and then estimate the patient's cardiac output in Liters per minute by dividing the amount of initial injection by this area.

Seconds after injection (t)	Dye concentration (adjusted for recirculation) (c)
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0

Unequal Subintervals

Consider the following table of values, whose subintervals are of unequal width. Compute the Left Riemann and Right Riemann sums as well as the Trapezoidal approximation.

x	0	2	3	7	9
$f(x)$	3	6	7	6	8

AP TIME

2001 AB2 (Parts a, b, and c)

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

2001 AB2 (Parts a, b, and c)
Answers

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \ {}^\circ\text{C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \ {}^\circ\text{C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \ {}^\circ\text{C/day}$$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer (with units)} \end{cases}$

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1\,{}^\circ\text{C}$

$2 : \begin{cases} 1 : \text{trapezoidal method} \\ 1 : \text{answer} \end{cases}$

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549\ {}^\circ\text{C/day}$

This means that the temperature is decreasing at the rate of $0.549\ {}^\circ\text{C/day}$ when $t = 12$ days.

$2 : \begin{cases} 1 : P'(12) \ \text{(with or without units)} \\ 1 : \text{interpretation} \end{cases}$

AP TIME

2005 AB3 (Part b...sort of)

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

2005 AB3 (Part b…sort of)
Answers

(b) $\frac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$$

Average temperature $\approx \frac{1}{8}A = 75.6875°\text{C}$

$$3 : \begin{cases} 1 : \frac{1}{8}\int_0^8 T(x)\,dx \\ 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$$

P.S. The mean on this question was 1.76!!!

AP TIME

Revisiting 2007 AB5 (Parts c and d)

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\, dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\, dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\, dt$? Give a reason for your answer.

Revisiting 2007 AB5 (Parts c and d)
Answers

(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of $\mathrm{ft}^3/\mathrm{min}$ in part (b) and ft in part (c)	1 : units in (b) and (c)

Recall: The mean on this question was 2.48!!!