L’Hôpital’s Rule
and
Relative Rates of Growth
Why are limits used again?

…To describe the behavior of functions!

L’Hopital’s Rule is simply a TECHNIQUE for finding limits, particularly limits that seem nonexistent.

The strongest form of this rule states the following:

Suppose that \( f(a) = g(a) = 0 \)

\( f \) and \( g \) are differentiable on an open interval \( I \) containing \( a \)

\( g'(x) \neq 0 \) on \( I \) if \( x \neq a \)

Then…

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

If the latter limit exists.
Example 1

Estimate the limit graphically and then use L’Hopital’s Rule to find the limit.

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$$
Example 2

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - \frac{x}{2}}{x^2}$$
Let’s Explore a little something…

Consider the function \( f(x) = \frac{\sin x}{x} \)

(1) Use L’Hospital’s Rule to find \( \lim_{x \to 0} f'(x) \)

(2) Let 
\[
\begin{align*}
y_1 &= \sin x, \\
y_2 &= x, \\
y_3 &= \frac{y_1}{y_2}, \\
y_4 &= \frac{y_1'}{y_2'}
\end{align*}
\]

Explain how graphing \( y_3 \) and \( y_4 \) in the same viewing window provides support for l’Hopital’s rule in part 1.

(3) Let \( y_5 = y_3' \)

Graph \( y_3, y_4 \), and \( y_5 \) in the same viewing window. Based on what you see in the viewing window, make a statement about what l’Hopital’s Rule does NOT say.
Using L’Hôpital’s Rule with One-Sided Limits

Example 3

Evaluate \[ \lim_{x \to 0^+} \frac{\sin x}{x^2} \]

Evaluate \[ \lim_{x \to 0^-} \frac{\sin x}{x^2} \]
Indeterminate Forms

Example 4

Identify the indeterminate form and evaluate the limit using l’Hospital’s Rule. Support your answer graphically.

\[
\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}
\]
Indeterminate Forms

Example 5

\[ \lim_{x \to \infty} x \sin \frac{1}{x} \]

Identify the indeterminate form and evaluate the limit using l’Hôpital’s Rule. Support your answer graphically.
Indeterminate Forms

Example 6

\[\frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty\]

Find

\[
\lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right)
\]

Double LHR trouble here!!!
Indeterminate Forms

$1^\infty, 0^0, \infty^0$

Limits that lead to these indeterminate forms can sometimes be handled by taking logarithms first. We use l’Hopital’s Rule to find the limit of the logarithm and then exponentiate to reveal the original function’s behavior.

\[
\lim_{x \to a} \ln f(x) = L \implies \lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L
\]

Here $a$ can be finite or infinite.
**Indeterminate Forms**

**Example 7**

Find

\[
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x
\]

1\(^\infty\), 0\(^0\), \infty\(^0\)

\[
\ln f(x) = \ln \left(1 + \frac{1}{x}\right)^x
\]

**Step 1:** Take \(\ln\) of both sides

\[
\ln f(x) = x \ln \left(1 + \frac{1}{x}\right)
\]

**Step 2:** Apply props of logs

\[
\ln f(x) = x \ln \left(1 + \frac{1}{x}\right)
\]

Now, if we don’t rewrite the function as a rational function, we won’t be able to apply LHR.

\[
\ln f(x) = \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}}
\]

Rewritten…
Example 7 Continued

\[ \lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln \left( 1 + \frac{1}{x} \right)}{\frac{1}{x}} = 0 \]

**Step 3: Apply LHR**

\[ \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right) \left( -\frac{1}{x^2} \right) = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right) = 1 \]

\[ \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^1 = e \]

Applying the theorem …..
Indeterminate Forms

$1^\infty, 0^0, \infty^0$

Example 8

$$\lim_{x \to 0^+} x^x$$

Example 9

$$\lim_{x \to \infty} \frac{1}{x^x}$$
Consider the differential equation \( \frac{dy}{dx} = 1 - y \). Let \( y = f(x) \) be the particular solution to this differential equation with the initial condition \( f(1) = 0 \). For this particular solution, \( f(x) < 1 \) for all values of \( x \).

(a) Use Euler’s method, starting at \( x = 1 \) with two steps of equal size, to approximate \( f(0) \). Show the work that leads to your answer.

(b) Find \( \lim_{x \to 1} \frac{f(x)}{x^3 - 1} \). Show the work that leads to your answer.

(c) Find the particular solution \( y = f(x) \) to the differential equation \( \frac{dy}{dx} = 1 - y \) with the initial condition \( f(1) = 0 \).
2010 BC5 - Answers

(a) \[ f\left(\frac{1}{2}\right) = f(1) + \left(\frac{dy}{dx}\right)_{(1, 0)} \cdot \Delta x \]
\[ = 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2} \]
\[ f(0) \approx f\left(\frac{1}{2}\right) + \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{1}{2}\right)} \cdot \Delta x \]
\[ \approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4} \]

(b) Since \( f \) is differentiable at \( x = 1 \), \( f \) is continuous at \( x = 1 \). So,
\[ \lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} (x^3 - 1) \]
and we may apply L’Hospital’s Rule.
\[ \lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3} \]
The provided answer is:

\[ \frac{dy}{dx} = 1 - y \]

\[ \int \frac{1}{1 - y} \, dy = \int 1 \, dx \]

\[ -\ln|1 - y| = x + C \]

\[ -\ln 1 = 1 + C \Rightarrow C = -1 \]

\[ \ln|1 - y| = 1 - x \]

\[ |1 - y| = e^{1-x} \]

\[ f(x) = 1 - e^{1-x} \]

Note: max 2/5 [1-1-0-0-0] if no constant of integration
Note: 0/5 if no separation of variables
Using L’Hôpital’s Rule to Compare Growth Rates

Let \( f(x) \) and \( g(x) \) be positive for \( x \) sufficiently large.

1) \( f(x) \) grows faster than \( g(x) \) (and \( g(x) \) grows slower than \( f(x) \)) as \( x \to \infty \) if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or equivalently if} \quad \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0
\]

2) \( f(x) \) and \( g(x) \) grow at the same rate as \( x \to \infty \) if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0 \quad \text{L is finite here and NOT 0.}
\]
Some Examples

Example 1

Show that the function $e^x$ grows faster than $x^2$ as $x \to \infty$

Example 2

Show that $x$ grows at the same rate as $x + \sin x$ as $x \to \infty$